On the temperature dependence of the interaction-induced entanglement.

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Abstract

Both direct and indirect weak nonresonant interactions are shown to produce entanglement between two initially disentangled systems prepared as a tensor product of thermal states, provided the initial temperature is sufficiently low. Entanglement is determined by the Peres-Horodeckii criterion, which establishes that a composite state is entangled if its partial transpose is not positive. If the initial temperature of the thermal states is higher than an upper critical value T_{uc} the minimal eigenvalue of the partially transposed density matrix of the composite state remains positive in the course of the evolution. If the initial temperature of the thermal states is lower than a lower critical value $T_{lc} \leq T_{uc}$ the minimal eigenvalue of the partially transposed density matrix of the composite state becomes negative which means that entanglement develops. We calculate the lower bound T_{lb} for T_{lc} and show that the negativity of the composite state is negligibly small in the interval $T_{lb} < T < T_{uc}$. Therefore the lower bound temperature T_{lb} can be considered as *the* critical temperature for the generation of entanglement.

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I. INTRODUCTION

Efficient simulation of quantum dynamics on classical computers is hampered by the problem of scaling: the complexity of computation in quantum dynamics scales exponentially with the number of degrees of freedom [1]. The reason for this exponential growth is the entanglement of the degrees of freedom that is generated during the evolution. This problem is of a fundamental character: entanglement is viewed as one of the main peculiarities of the quantum dynamics as compared to its classical counterpart [2, 3]. Asking under what conditions entanglement is generated along the evolution of the quantum system is closely associated with the question of the quantum-classical transition [4, 5].

It is customary in quantum dynamical simulations to assume that the initial state of the composite system is factorized in the relevant local basis [6]. An important question is whether the product form is conserved along the evolution [7, 8]. The answer was generally found to be negative both for the pure [8, 9] and for the mixed state [9] dynamics.

A pure composite state is entangled if and only if it is not factorized in the local basis. For mixed states the situation is more complex [10]. For a bipartite composite system separability [11] is defined as a decomposition of the density matrix of the composite system in the form

$$\hat{\boldsymbol{\rho}}_{12} = \sum p_i \hat{\boldsymbol{\rho}}_1^i \otimes \hat{\boldsymbol{\rho}}_2^i, \tag{1}$$

where $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$ and $\hat{\boldsymbol{\rho}}_1$ and $\hat{\boldsymbol{\rho}}_2$ are density matrices on Hilbert spaces of the first and the second subsystem, respectively. Separable states exhibit only classical correlations. States that cannot be represented in the form (1) exhibit correlations that cannot be explained within any classical theory and are said to be entangled. There are two qualitatively different kinds of the mixed states entanglement [12]: free entanglement and bound entanglement. Free entanglement can be brought in a form useful for quantum information processing and bound entanglement is "useless" in this sense.

Separable states are not of the product form generally. Thus the important question remains, under what conditions the mixed state of the composite system evolving from the initial product (or generally separable) state develops entanglement along the evolution. If quantum correlations in the composite system do not develop during the evolution one may speculate that the dynamics of the composite system has classical character. A possible

practical implication is that this "separable dynamics" could be simulated efficiently on classical computers.

The dynamics of entanglement was investigated recently in various systems: the quantum Brownian particle [13], harmonic chain [14], two-qubits system interacting with the common harmonic bath [15], Jaynes-Cummings model [16], NMR [17], various spin systems [18, 19, 20], Morse oscillator coupled to the spin bath [21] and bipartite Gaussian states in quantum optics [22] to mention just some cases. The purpose of the present paper is to explore the temperature dependence of entanglement generation in the course of evolution of a bipartite state in the limit of weak coupling and nonresonant interaction between the parts. Under these limitations nondegenerate perturbation theory was applied to the calculation of the bipartite entanglement in the evolving composite system. We have considered two cases of interaction - (1) direct interaction, when two initially disentangled systems are brought into contact (Cf. Fig.1), and (2) indirect interaction, when two noninteracting and initially disentangled systems are brought into contact with the third party (Cf. Fig.5). In each case the initial state of the composite system was taken to be the product of the thermal states of the parts.

To establish quantum entanglement the Peres-Horodeckii criterion is employed [23, 24]. The Peres-Horodeckii criterion states that the bipartite system is entangled when the partially transposed density matrix of the system possesses a negative eigenvalue. The converse statement is generally not true: there exist inseparable states whose partially transposes are positive [25]. It is proved in Ref.[12] that states whose partially transposes are positive (PPT states in what follows) do not exhibit free entanglement. Therefore PPT states are either separable or bound entangled and as a consequence are not useful in quantum information processing. In the context of simulating a quantum composite system with classical computers, we are interested in the possibility of maintaining a separable form (Cf. Eq.(1)) during the evolution. We conjecture on the basis of Ref.[26], where it is proved that PPT density matrices of sufficiently small rank are separable, that for a state that remains PPT during the evolution separability can be obtained by embedding in a larger Hilbert space.

Applying the Peres-Horodeckii criterion to the case (1) we show that for sufficiently low initial temperature of the subsystems the interaction does induce entanglement unless the ground state of either one of the subsystem is invariant under the interaction. A lower and upper critical temperatures T_{lc} and T_{uc} exist such that if the composite system evolves

from the initial thermal state with temperature $T < T_{lc}$ the minimal eigenvalue of the partially transposed density matrix becomes negative in the course of the evolution and if $T > T_{uc}$ the minimal eigenvalue of the partially transposed density matrix stays positive. The lower bound T_{lb} of the lower critical temperatures T_{lc} was calculated in the limit of weak intersystem coupling and shown to be tight: the negativity of the composite state [27], which is a quantitative counterpart of the Peres-Horodeckii criterion and a measure of entanglement, is shown to be generally negligible for temperatures in the interval $T_{lb} < T < T_{uc}$. Therefore, according to the Peres-Horodeckii criterion, when $T < T_{lb}$ the composite system develops entanglement in the course of the evolution and when $T > T_{lb}$ the composite state remains PPT state.

The question addressed in case (2) of indirect coupling is what are the conditions on the interaction with the common bath and on the initial temperature of the states which cause entanglement of the noninteracting systems? Two scenarios with time scales separation are studied: (a) two "slow" noninteracting systems coupled to a "fast" third party (b) two "fast" noninteracting systems coupled to "slow" third party. Under some technical assumptions about the form of the interaction we find in both cases that for sufficiently low initial temperature of the noninteracting systems entanglement is induced by the interaction with the third party. We calculate the lower bound temperature T_{lb} in both cases of the time scales separation. In the system of two noninteracting spins, coupled to the common bath, the lower bound coincides with the T_{uc}.

In both cases (1) and (2) the evolution starts from an uncorrelated initial state of the composite system represented by the tensor product of thermal states of the subsystems involved. As a consequence, initially the eigenstates of the partially transposed density matrix of the composite state are nonnegative. The evolution under the interaction perturbs the initial state. The new eigenvalues of the partially transposed density matrix are calculated by the nondegenerate perturbation theory assuming the coupling is weak and the interaction is nonresonanant. The time dependence of the minimal eigenvalue is not analyzed in detail. As the time evolution of the density matrix is quasiperiodic the minimal eigenvalue of the partially transposed density matrix is also a quasiperiodic function of time. The interaction is said to induce entanglement if the minimal eigenvalue becomes negative in the course of the evolution.
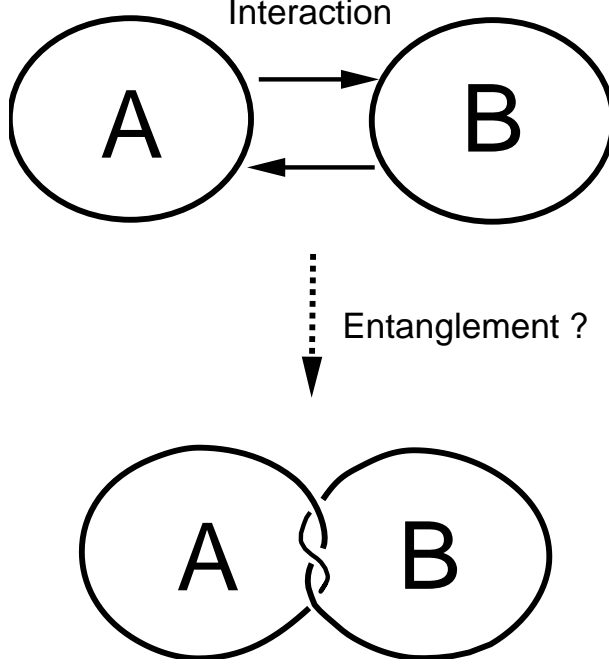
FIG. 1: The coupling scheme for two directly interacting systems.

II.   ENTANGLEMENT BETWEEN TWO DIRECTLY INTERACTING SYSTEMS

A composite system $A \otimes B$ evolves under the following Hamiltonian :

$$\hat{\mathbf{H}}_{total} \;=\; \hat{\mathbf{H}} + \gamma \hat{\mathbf{V}}. \tag{2}$$

where $\hat{\mathbf{H}} = \hat{\mathbf{H}}_a \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \hat{\mathbf{H}}_b$, $\hat{\mathbf{V}} = \hat{\mathbf{V}}_a \otimes \hat{\mathbf{V}}_b$ and γ scales the magnitude of the interaction. Let the initial state be

$$\hat{\boldsymbol{\rho}}(0) \;=\; \hat{\boldsymbol{\rho}}_a \otimes \hat{\boldsymbol{\rho}}_b, \tag{3}$$

where both $\hat{\boldsymbol{\rho}}_a$ and $\hat{\boldsymbol{\rho}}_b$ are thermal states: $\hat{\boldsymbol{\rho}}_{a,b} = Z_{a,b}^{-1} \exp(-\hat{\mathbf{H}}_{a,b}/T)$, where $Z_{a,b}^{-1}$ is the nor-malization factor. The Boltzman constant k_B is unity throughout the paper. The evolution is followed in the interaction picture. Then

$$\frac{\partial \hat{\boldsymbol{\rho}}'}{\partial t} \;=\; -i\gamma \left[\hat{\mathbf{V}}_a(t) \otimes \hat{\mathbf{V}}_b(t), \hat{\boldsymbol{\rho}}' \right], \tag{4}$$

where

$$\hat{\boldsymbol{\rho}}'(t) \;=\; e^{-i\hat{\mathbf{H}}t}\, \hat{\boldsymbol{\rho}}\, e^{i\hat{\mathbf{H}}t},$$
$$\hat{\mathbf{V}}_{a,b}(t) \;=\; e^{i\hat{\mathbf{H}}_{a,b}t} \hat{\mathbf{V}} e^{-i\hat{\mathbf{H}}_{a,b}t}. \tag{5}$$

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Here and in the rest of the paper we take $\hbar = 1$. It is clear that the density matrix $\hat{\boldsymbol{\rho}}(t)$ is separable if and only if $\hat{\boldsymbol{\rho}}'(t)$ separable. In what follows the tags in $\hat{\boldsymbol{\rho}}'(t)$ are omitted for simplicity. In the first order in the coupling γ the evolution of $\hat{\boldsymbol{\rho}}$ becomes:

$$\hat{\boldsymbol{\rho}}(t) = \hat{\boldsymbol{\rho}}(0) - i\gamma \int_0^t \left[\hat{\mathbf{V}}_a(t') \otimes \hat{\mathbf{V}}_b(t'), \hat{\boldsymbol{\rho}}(0) \right] dt'. \tag{6}$$

Entanglment of $\hat{\boldsymbol{\rho}}(t)$ is established by the application of the Peres-Horodeckii criterion. This is carried out by calculating the partial transpose of the state. The partial transposition T_a with respect to subsystem A of a bipartite state $\hat{\boldsymbol{\rho}}_{ab}$ expanded in a local orthonormal basis as $\hat{\boldsymbol{\rho}}_{ab} = \sum \rho_{ij,kl} \left| i \right\rangle \left\langle j \right| \otimes \left| k \right\rangle \left\langle l \right|$ is defined as:

$$\rho_{ab}^{T_a} \equiv \sum \rho_{ij,kl} \left| j \right\rangle \left\langle i \right| \otimes \left| k \right\rangle \left\langle l \right|. \tag{7}$$

The spectrum of the partially transposed density matrix does not depend on the choice of local basis or on the choice of the subsystem with respect to which the partial transposition is performed. By the Peres-Horodeckii criterion the eigenvalues of a partially transposed *separable* bipartite state are nonnegative.

The density operator (6) under the partial transposition (T^a) becomes:

$$\hat{\boldsymbol{\rho}}(t)^{T_a} = \hat{\boldsymbol{\rho}}(0)^{T_a} - i\gamma \int_0^t \left[\hat{\mathbf{V}}_a(t') \otimes \hat{\mathbf{V}}_b(t'), \hat{\boldsymbol{\rho}}(0) \right]^{T_a} dt'. \tag{8}$$

Let $\left| ik \right\rangle \equiv \left| i \right\rangle \otimes \left| k \right\rangle$ be the local orthonormal basis of the system $A \otimes B$ composed of the eigenvectors of the unperturbed Hamiltonian $\hat{\mathbf{H}} = \hat{\mathbf{H}}_a + \hat{\mathbf{H}}_b$:

$$\hat{\mathbf{H}}_{a,b} \left| i \right\rangle = E_{a,b}^i \left| i \right\rangle, \tag{9}$$

where $E_{a,b}^i$, $i = 1, 2, ...$, is the unperturbed energy spectrum of the Hamiltonian $\hat{\mathbf{H}}_{a,b}$. The initial state is of the tensor product form, Cf. Eq.(3), therefore :

$$\hat{\boldsymbol{\rho}}(0)^{T_a} \left| ik \right\rangle = \hat{\boldsymbol{\rho}}(0) \left| ik \right\rangle = \hat{\boldsymbol{\rho}}_a \otimes \hat{\boldsymbol{\rho}}_b \left| ik \right\rangle = P_{ik} \left| ik \right\rangle, \tag{10}$$

where $P_{ik} \equiv p_{a,i} p_{b,k}$ and $p_{a,i}, p_{b,k}$ are defined by $p_{a,i} = \left\langle i | \hat{\boldsymbol{\rho}}_a | i \right\rangle$ and $p_{b,k} = \left\langle k | \hat{\boldsymbol{\rho}}_b | k \right\rangle$. The matrix elements of $\hat{\boldsymbol{\rho}}(t)^{T_a}$ in the chosen basis are given by:

$$\left\langle ik \left| \hat{\boldsymbol{\rho}}(t)^{T_a} \right| jl \right\rangle = P_{ik} \delta_{(ik),(jl)} + M_{ik,jl}, \tag{11}$$

where

$$
\begin{aligned}
M_{ik,jl} &= i\gamma \int_0^t \left\langle ik \left| \left(\left[\hat{\mathbf{V}}_a(t') \otimes \hat{\mathbf{V}}_b(t'), \hat{\boldsymbol{\rho}}_a \otimes \hat{\boldsymbol{\rho}}_b \right] \right)^{T_a} \right| jl \right\rangle dt' \\
&= i\gamma \int_0^t \left(\left\langle ik \left| \hat{\boldsymbol{\rho}}_a \hat{\mathbf{V}}_a(t')^T \otimes \hat{\mathbf{V}}_b(t') \hat{\boldsymbol{\rho}}_b \right| jl \right\rangle \right. \\
&\quad - i\gamma \left\langle ik \left| \hat{\mathbf{V}}_a(t')^T \hat{\boldsymbol{\rho}}_a \otimes \hat{\boldsymbol{\rho}}_b \hat{\mathbf{V}}_b(t') \right| jl \right\rangle \right) dt' \\
&= i\gamma (P_{il} - P_{jk}) \int_0^t \left\langle i \left| \hat{\mathbf{V}}_a(t')^T \right| j \right\rangle \left\langle k \left| \hat{\mathbf{V}}_b(t') \right| l \right\rangle dt' \\
&= i\gamma (P_{il} - P_{jk}) \int_0^t \left\langle j \left| \hat{\mathbf{V}}_a \right| i \right\rangle \left\langle k \left| \hat{\mathbf{V}}_b \right| l \right\rangle e^{it'(E_a^i - E_a^j + E_b^l - E_b^k)} dt' \\
&= \gamma (P_{il} - P_{jk}) \left\langle j \left| \hat{\mathbf{V}}_a \right| i \right\rangle \left\langle k \left| \hat{\mathbf{V}}_b \right| l \right\rangle \frac{e^{it(E_a^i - E_a^j + E_b^l - E_b^k)} - 1}{(E_a^i - E_a^j + E_b^l - E_b^k)},
\end{aligned}
\tag{12}
$$

where $\hat{\mathbf{X}}^T$ designates the transpose of the operator $\hat{\mathbf{X}}$.

When $T = 0$, the zero eigenvalue of the initial state $\hat{\boldsymbol{\rho}}(0)$ is degenerate. As a result the zero eigenvalue of the partially transposed initial density operator $\hat{\boldsymbol{\rho}}(0)^{T_a} = \hat{\boldsymbol{\rho}}(0)$ is also degenerate. The zero eigenvalues correspond to empty initially unoccupied states. By the standard secular perturbation theory the first order correction to the degenerate eigenvalue $\lambda^{(0)} = 0$ of the matrix $\hat{\boldsymbol{\rho}}(0)^{T_a}$ is given by

$$
\left| M_{nn'} - \lambda^{(1)} \delta_{nn'} \right| = 0,
\tag{13}
$$

where $|n\rangle$ and $|n'\rangle$ are eigenvectors of the matrix $\hat{\boldsymbol{\rho}}_a^T \otimes \hat{\boldsymbol{\rho}}_b$, corresponding to the degenerate $\lambda^{(0)} = 0$. Since $\hat{\boldsymbol{\rho}}(0)^{T_a} = \hat{\boldsymbol{\rho}}_a^T \otimes \hat{\boldsymbol{\rho}}_b = \hat{\boldsymbol{\rho}}_a \otimes \hat{\boldsymbol{\rho}}_b$ the eigenvectors of $\hat{\boldsymbol{\rho}}_a^T \otimes \hat{\boldsymbol{\rho}}_b$, corresponding to $\lambda^{(0)} = 0$ are $|n\rangle = \{|1\rangle \otimes |i\rangle, |i\rangle \otimes |1\rangle, |i\rangle \otimes |j\rangle, i, j = 2, 3, ..\}$.

Therefore at $T = 0$, $P_{ik} = \delta_{ik}\delta_{k1}$, and by inspection of Eq. (12), the only nonvanishing matrix elements in the degenerate subspace spanned by $|n\rangle$ and $|n'\rangle$ are $M_{1i,j1}$ and $M_{j1,1i}$ where either $i \neq 1$ or $j \neq 1$. Since the trace of the matrix M is zero, either all its eigenvalues vanish or some of them are negative. All the eigenvalues of M cannot vanish unless $M = 0$, which from Eq. (12) implies $\left[\hat{\mathbf{V}}_a, \hat{\boldsymbol{\rho}}_a\right] = 0$ or $\left[\hat{\mathbf{V}}_b, \hat{\boldsymbol{\rho}}_b\right] = 0$, i.e. the ground state of either one of the subsystem is invariant under the interaction. In this case the interaction acts locally on the subsystems and cannot entangle them. Otherwise there are negative solutions to Eq. (13) and as a consequence the partial transpose of the density operator attains negative eigenvalues already in the first order in the coupling. Therefore, according to the Peres-Horodeckii criterion, entanglement develops at zero temperature.

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To simplify the study of the generation of entanglement at finite temperatures it is assumed that the only non zero matrix elements of $\hat{\mathbf{V}}_{a,b}$ are those between neighboring states, i.e. $(\hat{\mathbf{V}}_{a,b})_{ij} \propto \delta_{i,j\pm1}$. Under this assumption the partially transposed density matrix $\hat{\boldsymbol{\rho}}(t)^{T_a}$ obtains the following structure:

$$\hat{\boldsymbol{\rho}}(t)^{T_a} = \begin{pmatrix} P_{11} & 0 & 0 & M_{11,22} & 0 & 0 & 0 & . & . \\ 0 & P_{12} & M_{12,21} & 0 & M_{12,23} & 0 & 0 & . & . \\ 0 & M_{12,21}^* & P_{21} & 0 & 0 & M_{21,32} & 0 & . & . \\ M_{11,22}^* & 0 & 0 & P_{22} & 0 & 0 & M_{22,33} & . & . \\ 0 & M_{12,23}^* & 0 & 0 & P_{23} & M_{23,32} & 0 & . & . \\ 0 & 0 & M_{21,32}^* & 0 & M_{23,32}^* & P_{32} & 0 & . & . \\ 0 & 0 & 0 & M_{22,33}^* & 0 & 0 & P_{33} & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix}, \qquad (14)$$

where P_{ij} are defined after the Eq.(10) and $M_{ki,jl}$ by Eq.(12).

There are two kinds of matrix elements $M_{ki,jl}$: $M_{ki,(k+1)(i+1)}$ and $M_{ki,(k\pm1)(i\mp1)}$ (other elements are their counterparts under the transposition). Matrix elements $M_{ki,(k+1)(i+1)}$ couple the unperturbed eigenvalues P_{ki} and $P_{(k+1)(i+1)}$. For small coupling strength $\gamma \left|M_{ki,(k+1)(i+1)}\right| \ll P_{ki}$ and the contribution of $M_{ki,(k+1)(i+1)}$ to the correction to P_{ki} is negligible and cannot make the eigenvalue negative. On the other hand, the ratio $\left|M_{ki,(k+1)(i+1)}\right|/P_{(k+1)(i+1)} \propto \gamma(P_{k(i+1)} - P_{(k+1)i})/P_{(k+1)(i+1)}$ can in general be arbitrary large for low temperatures but for sufficiently high temperatures it tends to zero and as a consequence the contribution of the coupling element $M_{ki,(k+1)(i+1)}$ to the correction to $P_{(k+1)(i+1)}$ is negligible. It can be checked along the same lines that the ratio of the coupling matrix elements $M_{ki,(k\pm1)(i\mp1)}$ to the unperturbed eigenvalues P_{ki} and $P_{(k\pm1)(i\mp1)}$ of the partially transposed density matrix (14) vanish for sufficiently high temperature. Therefore, at least for composite systems with finite Hilbert space dimensions, there exists a finite upper critical temperature T_{uc}. Above T_{uc} the spectrum of the partially transposed density matrix remains positive (PPT). In close vicinity of T_{uc} from below the minimal eigenvalue becomes negative in the course of the evolution. These conclusions stay in accord with a general result [28, 29] that finite dimensional composite states in sufficiently small neighbourhood of the maximally mixed state (i.e. thermal states at infinite temperature) are separable. We conjecture, that for an infinite composite system, the upper critical temperature T_{uc} exists

if the energy spacing is bound.

At sufficiently low initial temperature the minimal eigenvalue of the partially transposed density matrix becomes negative in the course of the evolution. This means that there exists a finite lower critical temperature T_{lc}. Below T_{lc} the composite systems $A \otimes B$ develops entanglement. In sufficiently close vicinity of T_{lc} from above the state remains PPT in the course of evolution. It is possible that $T_{lc} = T_{uc}$. This equality is confirmed in all numerical tests. A lower bound T_{lb} for the lower critical temperature T_{lc} can be calculated using perturbation analysis. It is shown that this bound is tight since the free entanglement in the interval $T_{lb} < T < T_{uc}$ is negligibly small under the weak coupling assumption. Therefore, from the practical point of view the lower bound T_{lb} for T_{lc} can be considered as *the* critical temperature for entanglement. For simplicity the lower bound for the lower critical temperature is termed "the lower bound temperature" throughout the paper.

At low temperatures the leading order contribution to the negative eigenvalue of the partially transposed density matrix comes from the matrix elements $M_{11,22}$, $M_{12,21}$ (and their complex conjugates) that do not vanish at $T = 0$. Therefore, to the leading order in γ, the nonvanishing eigenvalues of the partially transposed density matrix Eq.(14) are the eigenvalues of the following effective partially transposed density matrix $\hat{\boldsymbol{\rho}}(t)_{eff}^{T_a}$:

$$
\hat{\boldsymbol{\rho}}(t)_{eff}^{T_a} = \begin{pmatrix} P_{11} & 0 & 0 & M_{11,22} \\ 0 & P_{12} & M_{12,21} & 0 \\ 0 & M_{12,21}^* & P_{21} & 0 \\ M_{11,22}^* & 0 & 0 & P_{22} \end{pmatrix}.
\tag{15}
$$

The critical temperature, calculated for the effective 4×4 matrix (15), is a lower bound for the lower critical temperature T_{lc} of the bipartite system $A \otimes B$. The eigenvalues of Eq. (15) are eigenvalues of two 2×2 matrices:

$$
\begin{pmatrix} P_{12} & M_{12,21} \\ M_{12,21}^* & P_{21} \end{pmatrix}
\tag{16}
$$

and

$$
\begin{pmatrix} P_{11} & M_{11,22} \\ M_{11,22}^* & P_{22} \end{pmatrix}.
\tag{17}
$$

The eigenvalues of the matrix (16) are:

$$
\lambda_{\pm} = \frac{P_{12} + P_{21} \pm \sqrt{(P_{12} + P_{21})^2 - 4(P_{12}P_{21} - |M_{12,21}|^2)}}{2},
\tag{18}
$$

where from Eq.(12):

$$M_{12,21} = \gamma \left\langle 2 \left| \hat{\mathbf{V}}_a \right| 1 \right\rangle \left\langle 2 \left| \hat{\mathbf{V}}_b \right| 1 \right\rangle \frac{e^{it\Delta E_{11}} - 1}{\Delta E_{11}} (P_{11} - P_{22}), \tag{19}$$

where we define $\Delta E_{11} = E_a^2 - E_a^1 + E_b^2 - E_b^1$, which is the lowest joint excitation energy of the composite system.

From Eq. (18), λ_- will be negative whenever $P_{12}P_{21} < |M_{12,21}|^2$ and positive if $P_{12}P_{21} > |M_{12,21}|^2$. The lower bound temperature T_{lb} is evaluated from the condition $P_{12}P_{21} = |M_{12,21}|^2$. Since $|M_{12,21}|$ is an oscillating function of time (Cf. Eq. (19)) the amplitude of $|M_{12,21}|$ is taken to be equal to $\sqrt{P_{12}P_{21}}$:

$$\frac{2\gamma}{\Delta E_{11}} \left| \left\langle 2 \left| \hat{\mathbf{V}}_a \right| 1 \right\rangle \left\langle 2 \left| \hat{\mathbf{V}}_b \right| 1 \right\rangle \right| (P_{11} - P_{22}) = \sqrt{P_{12}P_{21}}. \tag{20}$$

Assuming that T_{lb} is low $P_{11} - P_{22} \approx P_{11}$ and then

$$\frac{2\gamma}{\Delta E_{11}} \left| \left\langle 2 \left| \hat{\mathbf{V}}_a \right| 1 \right\rangle \left\langle 2 \left| \hat{\mathbf{V}}_b \right| 1 \right\rangle \right| = \sqrt{\frac{P_2 P_3}{P_1^2}} = e^{-\frac{\Delta E_{11}}{2T_{lb}}}. \tag{21}$$

Since $e^{-\frac{\Delta E_{11}}{2T}}$ is a monotonic function of the temperature, at $T > T_{lb}$ $\lambda_- > 0$ and at $T < T_{lb}$ $\lambda_- < 0$. Finally, the expression for the lower bound temperature T_{lb} becomes:

$$T_{lb} = -\frac{\Delta E_{11}}{2 \ln \left(\frac{2\gamma}{\Delta E_{11}} \left| \left\langle 2 \left| \hat{\mathbf{V}}_a \right| 1 \right\rangle \left\langle 2 \left| \hat{\mathbf{V}}_b \right| 1 \right\rangle \right| \right)}. \tag{22}$$

So far only two of the eigenvalues of the matrix (15) have been evaluated. The other two eigenvalues are found to be strictly positive at and above the temperature T_{lb}. Therefore, the expression (22) defines the critical temperature for the partially transposed effective density matrix (15) and the lower bound temperature of the partially transposed density matrix (14).

Eq. (22) can be generalized to an interaction term of the form $\sum \gamma_i \hat{\mathbf{V}}_a^i \otimes \hat{\mathbf{V}}_b^i$:

$$T_{lb} = -\frac{\Delta E_{11}}{2 \ln \left(\frac{2}{\Delta E_{11}} \left| \sum_i \gamma_i \left\langle 2 \left| \hat{\mathbf{V}}_a^i \right| 1 \right\rangle \left\langle 2 \left| \hat{\mathbf{V}}_b^i \right| 1 \right\rangle \right| \right)}, \tag{23}$$

provided $\sum_i \gamma_i \left\langle 2 \left| \hat{\mathbf{V}}_a^i \right| 1 \right\rangle \left\langle 2 \left| \hat{\mathbf{V}}_b^i \right| 1 \right\rangle \neq 0$. When this term vanishes there is no entanglement in the first order in the coupling strength γ.

For the system of two interacting spins the lower bound T_{lb} given by Eq. (22) coincides with the upper critical temperature T_{uc} therefore in this case *the* critical temperature exists
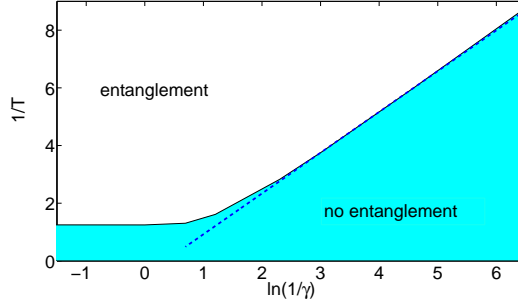
FIG. 2: The shaded area in the parameter space of the inverse initial temperature T of two spins and the logarithm of the inverse coupling strength γ, represents values of T and γ, where entanglement does not develop in the course of the evolution. The composite system of two spins evolves from the initial product of thermal states under the Hamiltonian $\hat{\mathbf{H}} = \frac{1}{2}\omega(\hat{\boldsymbol{\sigma}}_z^a \otimes \hat{\mathbf{1}} + (\sqrt{2}-1)\hat{\mathbf{1}} \otimes \hat{\boldsymbol{\sigma}}_z^b) + \gamma(\hat{\boldsymbol{\sigma}}_x^a \otimes \hat{\boldsymbol{\sigma}}_x^b - \hat{\boldsymbol{\sigma}}_y^a \otimes \hat{\boldsymbol{\sigma}}_y^b)$. The evolution is calculated numerically for $\omega = 1$. The border of the shaded area represents T_{uc} calculated numerically. The dashed line represents T_{lb} according to Eq. (22). Up to the coupling $\gamma = 0.1$ T_{lb} approximates T_{lc} very well.

in the strict sense. Fig. 2 shows results of the numerical calculation of the critical temperature as a function of coupling strength for a system of two interacting spins evolving from the initial product state of two thermal states. The Peres-Horodeckii criterion was used and the partial transpose of the evolving density matrix was calculated numerically to determine entanglement. The shaded area in the parametric space of the logarithm of inverse coupling and the inverse initial temperature represents the values of the parameters where no entanglement develops. For coupling up to $\gamma = 0.1$ T_{lb} given by Eq. (22) (the dashed line) corresponds well to the numerical values of T_{uc}. It is interesting to note that for large values of coupling the critical temperature asymptotically tends to a finite constant value of the same order of magnitude as the energy difference between the first excited and the ground state of the unperturbed composite system.

At $T < T_{lb}$ the minimal eigenvalue of the partially transposed state (14) is negative. We want to show that above T_{lb} the negative eigenvalues of the matrix (14) are of higher order in γ and therefore are negligibly small when the coupling is weak.

Let's consider corrections to the eigenvalues $P_{i(j+1)}$ and $P_{(i+1)j}$ of the composite state (14). The order of magnitude estimate of the smallest one of the corrected eigenvalues is :
$\lambda_-^{ij} = \sqrt{P_{i(j+1)}P_{(i+1)j}} - \gamma P_{ij}/\Delta E_{ij}$, where $\Delta E_{ij} \equiv E_a^{i+1} - E_a^i + E_b^{j+1} - E_b^j$. For simplicity

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we assume $P_{i(j+1)} = P_{(i+1)j}$. Then $\lambda_-^{ij} = O(P_{i(j+1)} - \gamma P_{ij}/\Delta E_{ij})$. Below T_{lb} the minimal eigenvalue of the state (14) is $\lambda_- = O(-\gamma/\Delta E_{11})$. We shall estimate the ratio $r^{ij} \equiv \lambda_-^{ij}/\lambda_-$ and show that it is negligible when the coupling is weak. We shall assume without loss of generality that the ground state energy is zero: $E_a^1 + E_b^1 = 0$. Then the partition function Z of the composite system is larger than unity. It follows that

$$r^{ij} = \frac{\lambda_-^{ij}}{\lambda_-} = \frac{\frac{\gamma}{\Delta E_{ij}}P_{ij} - P_{i(j+1)}}{\frac{\gamma}{\Delta E_{11}}} < \frac{\frac{\gamma}{\Delta E_{ij}}ZP_{ij} - ZP_{i(j+1)}}{\frac{\gamma}{\Delta E_{11}}} = \frac{\frac{\gamma}{\Delta E_{ij}}e^{-E_{ij}/T} - e^{-E_{i(j+1)}/T}}{\frac{\gamma}{\Delta E_{11}}}. \quad (24)$$

We are looking for the maximal value of r^{ij} in the interval $0 < T < T_c^{ij}$, corresponding to the condition $P_{i(j+1)} < \gamma P_{ij}/\Delta E_{ij}$, i.e. to the negative values of λ_-^{ij}. T_c^{ij} is determined by the condition $\lambda_-^{ij} = 0$. The ratio r^{ij} is positive in the interval $0 < T < T_c^{ij}$ and vanishes on its borders. Therefore r^{ij} has a maximum r_m^{ij} at $0 < T_m^{ij} < T_c^{ij}$, which is found from the condition $\partial r^{ij}/\partial T|_{T_m^{ij}} = 0$. The calculation gives $\exp(-\Delta E_{ij}/(2T_m^{ij})) = (\gamma/\Delta E_{ij})(E_{ij}/E_{i(j+1)}) < (\gamma/\Delta E_{ij}) = \exp(-\Delta E_{ij}/(2T_c^{ij}))$, which proves that there is one maximum r_m^{ij} at $0 < T_m^{ij} < T_c^{ij}$. We remark, that T_c^{ij}, corresponding to the largest ΔE_{ij} over all i and j, T_{uc}^*, is of the order of the upper critial temperature $T_{uc}^* = O(T_{uc})$. The maximal value of r^{ij} is given by:

$$r_m^{ij} = \frac{\Delta E_{11}}{2E_{ij} + \Delta E_{ij}} \left(\frac{2E_{ij}}{2E_{ij} + \Delta E_{ij}}\right)^{\frac{2E_{ij}}{\Delta E_{ij}}} \left(\frac{\gamma}{\Delta E_{ij}}\right)^{\frac{2E_{ij}}{\Delta E_{ij}}} < \frac{\Delta E_{11}}{2E_{ij} + \Delta E_{ij}} \left(\frac{\gamma}{\Delta E_{ij}}\right)^{\frac{2E_{ij}}{\Delta E_{ij}}}, \quad (25)$$

where the inequality follows from the fact that $1/e < \left(\frac{2E_{ij}}{2E_{ij} + \Delta E_{ij}}\right)^{\frac{2E_{ij}}{\Delta E_{ij}}} < 1$ in general. As a next step we notice that $\Delta E_{11} \leq 2E_{ij}$, therefore

$$r^{ij} < r_m^{ij} < \frac{\Delta E_{11}}{2E_{ij} + \Delta E_{ij}} \left(\frac{\gamma}{\Delta E_{ij}}\right)^{\frac{2E_{ij}}{\Delta E_{ij}}} \leq \frac{\Delta E_{11}}{\Delta E_{11} + \Delta E_{ij}} \left(\frac{\gamma}{\Delta E_{ij}}\right)^{\frac{\Delta E_{11}}{\Delta E_{ij}}}. \quad (26)$$

Introducing the definition $x_{ij} \equiv \Delta E_{ij}/\Delta E_{11}$ and taking $\Delta E_{11} = 1$, which corresponds to a rescaling of the coupling strength γ, leads to:

$$\frac{\lambda_-^{ij}}{\lambda_-} \equiv r^{ij} < \frac{1}{x_{ij}} \left(\frac{\gamma}{x_{ij}}\right)^{\frac{1}{x_{ij}}}. \quad (27)$$

Typically the spectrum becomes denser with increasing energy. In that case $x_{ij} \equiv \Delta E_{ij}/\Delta E_{11} \leq 1$. Values of λ_-^{ij}, corresponding to $x_{ij} \ll 1$ need not be taken into account,

because $T_c^{ij} < T_{lb}$ in this case and as a conseqence $\lambda_-^{ij} > 0$ at $T \geq T_{lb}$. At $x_{ij} = O(1)$ the upper bound for r scales as $O(\gamma)$ and therefore corresponding negative eigenvalues of Eq. (14) are negligible. In this case we expect that $T_{lb} \approx T_{uc}$.

In those cases when $x_{ij} \equiv \Delta E_{ij}/\Delta E_{11} \gg 1$ the upper bound for r scales as $O(1/x_{ij})$ and the corresponding negative eigenvalues of Eq. (14) can be neglected, too.

When x_{ij} is moderately larger than unity the upper bound Eq.(27) for r^{ij} has a local maximum. The position of the maximum weakly depends on γ: numerical calculations show $x_{ij} \approx 2 - 10$ in the range of $10^{-4} \leq \gamma \leq 10^{-1}$. The value of the minimum is a monotonically slowly increasing function of γ. In the range $10^{-4} \leq \gamma \leq 10^{-1}$ numerical estimation of Eq.(27) shows values $0.04 - 0.1$ for the local maximum. It is clear that the upper bound Eq.(27) for r^{ij} is far from being tight. In fact, numerical calculations show that r^{ij} is generally much smaller. As a consequence, the corresponding negative eigenvalues of Eq. (14) can be neglected.

It can be argued that although each one of the negative eigenvalues of Eq. (14) is negligible at $T \geq T_{lb}$ the (free) entanglement of the state cannot be neglected. In fact, the minimal negative eigenvalue of the partially transposed matrix is not a measure of entanglement. Various measures of entanglement have been defined [30]. In the present context we will employ a quantitative counterpart of the Peres-Horodeckii criterion, the negativity [27]:

$$N(\hat{\boldsymbol{\rho}}(t)) \equiv \frac{\left\|\hat{\boldsymbol{\rho}}(t)^{T_a}\right\| - 1}{2},\tag{28}$$

where $\left\|\hat{\mathbf{X}}\right\| = Tr\sqrt{\hat{\mathbf{X}}^\dagger \hat{\mathbf{X}}}$ is the trace norm of an operator $\hat{\mathbf{X}}$. The negativity of the state equals the absolute value of the sum of the negative eigenvalues of the partially transposed state. When the negativity of a composite bipartite state vanishes there is no free entanglement in the state. It can be shown by the order of magnitude analysis similar to the analysis above that values of the negativity of the composite state, corresponding to the partial transpose (14), are generally dominated by the minimal negative eigenvalue. As a consequence, the negativity of the state, evolving from the initial thermal product state at the temperature $T \geq T_{lb}$, is negligible under the weak coupling assumption.

Figures 3 and 4 display results of numerical calculations of the time averaged negativity of the composite state (6) as a function of initial temperature for two different kinds of unperturbed spectra of the composite system $A \otimes B$. Both A and B are four level systems. The composite system evolves from the initial product of thermal states of A and B under
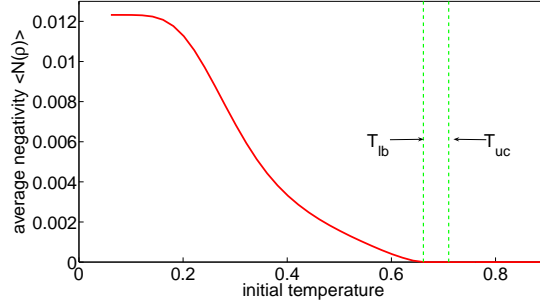
FIG. 3: The time averaged negativity as a function of initial temperature. The composite system is constructed from two interacting four level subsystems. The initial state is a product of thermal states. The evolution is generated numerically by the Hamiltonian (2) (for details of the Hamiltonian see the text) with $\gamma = 0.05$. The dashed lines correspond to the lower bound temperature T_{lb}, Eq. (22), and to the upper critical temperature T_{uc}, found numerically. It can be seen that the entanglement is vanishingly small in the interval $T_{lb} < T < T_{uc}$.

the Hamiltonian (2).

Fig. 3 presents the results of calculations for the following choice of the unperturbed spectra of $\hat{\mathbf{H}}_a$ and $\hat{\mathbf{H}}_b$: $E_a^{\{1,2,3,4\}} = \{1, 5, 8, 10\}$ and $E_b^i = \sqrt{E_a^i}$. Care was taken to avoid resonances and the spectra were chosen to become denser with increasing energy. The interaction terms in the Hamiltonian were restricted to $(\hat{\mathbf{V}}_{a,b})_{ij} = \delta_{i,j\pm1}$ and the coupling strength $\gamma = 0.05$. We see that $T_{uc} \approx T_{lb}$ and the time averaged negativity $\langle N(\hat{\boldsymbol{\rho}}(t)) \rangle$ is negligible in the interval $T_{lb} < T < T_{uc}$ as expected.

Fig. 4 displays the time averaged negativity $\langle N(\hat{\boldsymbol{\rho}}(t)) \rangle$ as a function of initial temperature of the composite state of two interacting four level subsystems A and B with the unperturbed energy spectra $E_a^{\{1,2,3,4\}} = \{1, 3, 7, 13\}$ and $E_b^i = \sqrt{E_a^i}$. The composite state evolves from the initial product of two thermal states under the Hamiltonian (2), where $(\hat{\mathbf{V}}_{a,b})_{ij} = \delta_{i,j\pm1}$ and the coupling strength $\gamma = 0.05$. In choosing the unperturbed spectra care was taken to avoid resonances and to ensure that the maximal value of $x_{ij} \equiv \Delta E_{ij}/\Delta E_{11}$ equals the position of the local maximum of the upper bound (27), corresponding to $\gamma = 0.05$. Fig. 3 shows that the time averaged negativity $\langle N(\hat{\boldsymbol{\rho}}(t)) \rangle$ is negligible in the interval $T_{lb} < T < T_{uc}$ as expected. The value of $T_c^{ij} \equiv T_{uc}^*$ (the definition of T_c^{ij} is given after Eq.(24)), corresponding to the maximal value $\Delta E_{max} \equiv \max_{ij}(\Delta E_{ij})$ is calculated. T_{uc}^* is in good correspondence with the value T_{uc}, calculated numerically.
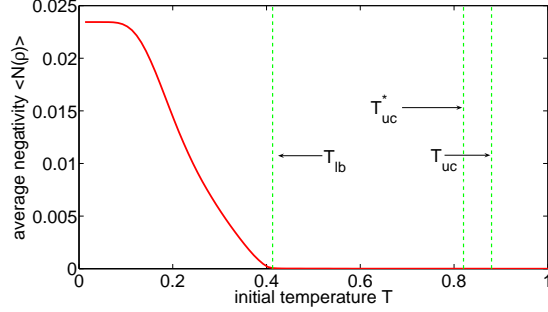
FIG. 4: The time averaged negativity as a function of initial temperature of the composite system. The composite system is constructed from two interacting four level subsystems. The initial state is a product of thermal states. The evolution is generated numerically by the Hamiltonian (2) (for details of the Hamiltonian see the text) with $\gamma = 0.05$. The dashed lines correspond to the lower bound temperature T_{lb} Eq. (22), to the numerical value of the upper critical temperature T_{uc} and to the value T_{uc}^*, corresponding to the largest spectrum spacing ΔE_{max}. We see that entanglement is vanishingly small at $T_{lb} < T < T_{uc}$, as expected, and T_{uc}^* is a good approximation to the upper critical temperature T_{uc}.

III.   ENTANGLEMENT BETWEEN TWO NONINTERACTING SYSTEMS IN CONTACT WITH A COMMON THIRD PARTY

The dynamics studied is of the composite system $A \otimes B \otimes C$ where systems B and C do not interact directly (Cf. Fig.5). The entanglement explored is of the reduced composite system $B \otimes C$.

The evolution is generated by the following Hamiltonian:

$$\hat{\mathbf{H}}_{total} = \hat{\mathbf{H}} + \gamma\hat{\mathbf{V}}, \tag{29}$$

where $\hat{\mathbf{H}} = \hat{\mathbf{H}}_a + \hat{\mathbf{H}}_b + \hat{\mathbf{H}}_c$ and $\hat{\mathbf{V}} = \gamma\hat{\mathbf{V}}_a \otimes (\hat{\mathbf{V}}_b \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \hat{\mathbf{V}}_c)$. The analysis is carried out in the interaction picture. The initial state is taken to be $\hat{\boldsymbol{\rho}}(0) = \hat{\boldsymbol{\rho}}_a \otimes \hat{\boldsymbol{\rho}}_b \otimes \hat{\boldsymbol{\rho}}_c$, where $\hat{\boldsymbol{\rho}}_a$, $\hat{\boldsymbol{\rho}}_b$ and $\hat{\boldsymbol{\rho}}_c$ are thermal states. Since B and C are noninteracting entanglement will appear only in the second order in the coupling. Up to second order in γ the state of the composite system $A \otimes B \otimes C$ becomes:
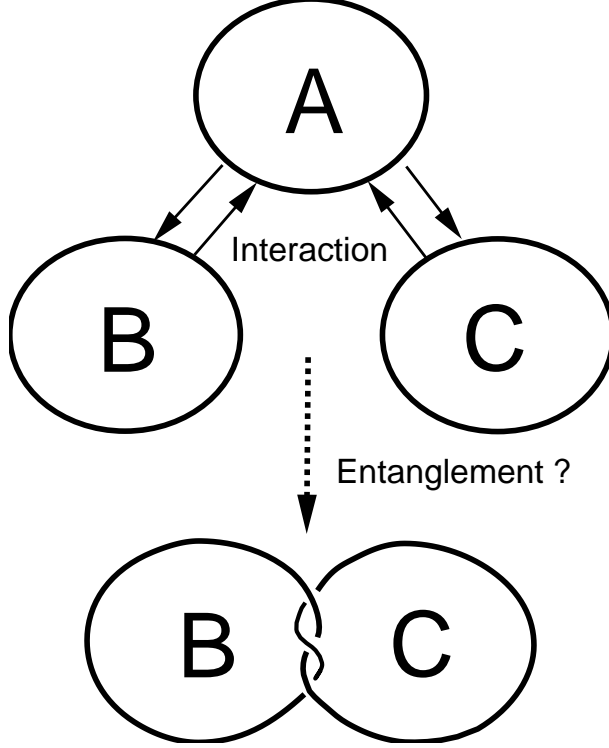
FIG. 5: Scheme of interaction for two noninteracting systems in contact with a common third party.

$$\hat{\boldsymbol{\rho}}(t)' = \hat{\boldsymbol{\rho}}(0) - i\gamma \int_0^t \left[\hat{\mathbf{V}}(t'), \hat{\boldsymbol{\rho}}(0)' \right] dt'$$
$$- \gamma^2 \int_0^t \int_0^{t'} \left[\hat{\mathbf{V}}(t'), \left[\hat{\mathbf{V}}(t''), \hat{\boldsymbol{\rho}}(0)' \right] \right] dt' dt'', \tag{30}$$

where

$$\hat{\boldsymbol{\rho}}' = e^{-i\hat{\mathbf{H}}t} \hat{\boldsymbol{\rho}} \, e^{i\hat{\mathbf{H}}t},$$
$$\hat{\mathbf{V}}(t) = e^{i\hat{\mathbf{H}}t} \hat{\mathbf{V}} e^{-i\hat{\mathbf{H}}t}. \tag{31}$$

In what follows the tag above the $\hat{\boldsymbol{\rho}}(t)$ is omitted.

Next the system is reduced to $B \otimes C$ by taking the partial trace of $\hat{\boldsymbol{\rho}}(t)$ over the system A degrees of freedom and the partial transposition with respect to the subsystem B is taken:

$$\hat{\boldsymbol{\rho}}_{bc}^{T_b}(t) = \hat{\boldsymbol{\rho}}_{bc}^{T_b}(0) + \hat{\mathbf{M}}(t), \tag{32}$$

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where $\hat{\boldsymbol{\rho}}_{bc}(t) \equiv Tr_a(\hat{\boldsymbol{\rho}}(t))$ and

$$\hat{\mathbf{M}} \equiv -i\gamma \int_0^t Tr_a\left(\left[\hat{\mathbf{V}}(t'), \hat{\boldsymbol{\rho}}(0)\right]\right)^{T_b} dt'$$
$$- \gamma^2 \int_0^t \int_0^{t'} Tr_a\left(\left[\hat{\mathbf{V}}(t'), \left[\hat{\mathbf{V}}(t''), \hat{\boldsymbol{\rho}}(0)\right]\right]\right)^{T_b} dt'dt''. \tag{33}$$

Let $|ik\rangle \equiv |i\rangle \otimes |k\rangle$ be the local orthonormal basis of the system $B \otimes C$ composed of the eigenstates of the Hamiltonian $\hat{\mathbf{H}}_b + \hat{\mathbf{H}}_c$:

$$\hat{\mathbf{H}}_{b,c} |i\rangle = E_{b,c}^i |i\rangle, \tag{34}$$

where $E_{b,c}^i$, $i = 1, 2, ...$, is the unperturbed energy spectrum of the Hamiltonian $\hat{\mathbf{H}}_{b,c}$. Since $\hat{\boldsymbol{\rho}}_{bc}(0) = \hat{\boldsymbol{\rho}}_b \otimes \hat{\boldsymbol{\rho}}_c$:

$$\hat{\boldsymbol{\rho}}_{bc}(0)^{T_b} |ik\rangle = \hat{\boldsymbol{\rho}}_{bc}(0) |ik\rangle = \hat{\boldsymbol{\rho}}_b \otimes \hat{\boldsymbol{\rho}}_c |ik\rangle = P_{ik} |ik\rangle, \tag{35}$$

where $P_{ik} \equiv p_{b,i}p_{c,k}$, and $p_{a,i}, p_{b,k}$ are defined by $p_{b,i} = \langle i|\hat{\boldsymbol{\rho}}_a|i\rangle$ and $p_{c,k} = \langle k|\hat{\boldsymbol{\rho}}_b|k\rangle$. The matrix elements of $\hat{\boldsymbol{\rho}}_{bc}(t)^{T_b}$ are given by:

$$\langle ik |\hat{\boldsymbol{\rho}}_{bc}(t)^{T_b}| jl\rangle = P_{ik}\delta_{(ik),(jl)} + M_{ik,jl}, \tag{36}$$

where by definition $M_{ik,jl} = \langle ik |\hat{\mathbf{M}}| jl\rangle$.

From this point the calculations proceed along the same lines as in Section II following Eq.(11). The minimal eigenvalue of the partially transposed reduced state $\hat{\boldsymbol{\rho}}_{bc}(t)^{T_b}$ is shown to be negative at sufficiently low temperatures and the lower bound temperature T_{lb} is calculated.

The negative eigenvalue of the partially transposed composite state Eq.(32) is calculated to the leading order in the coupling strength γ assuming $\langle n_i |\hat{\mathbf{V}}_i| m_i\rangle \propto \delta_{n_i \, m_i \pm 1}$. As in Section II the eigenvalue is found from the spectrum of the 2×2 matrix:

$$\begin{pmatrix} P_{12} + M_{12,12} & M_{12,21} \\ M_{12,21}^* & P_{21} + M_{21,21} \end{pmatrix}, \tag{37}$$

completely analogous to the matrix (16). The eigenvalues of Eq. (37) are:

$$\lambda_{\pm} = \frac{P_{12} + M_{12,12} + P_{21} + M_{21,21}}{2} \tag{38}$$
$$\pm \frac{\sqrt{(P_{12} + M_{12,12} + P_{21} + M_{21,21})^2 - 4((P_{12} + M_{12,12})(P_{21} + M_{21,21}) - |M_{12,21}|^2)}}{2}$$

and the eigenvalue λ_- becomes negative when $(P_{12} + M_{12,12})(P_{21} + M_{21,21}) < |M_{12,21}|^2$.

To calculate $M_{12,12}$, $M_{21,21}$ and $M_{12,21}$ we first note that the integrand in the first order term in Eq. (33) is:

$$
\begin{aligned}
Tr_a \left(\left[\hat{\mathbf{V}}(t'), \hat{\boldsymbol{\rho}}(0) \right] \right)^{T_b} &= \left\langle \hat{\mathbf{V}}_a \right\rangle \left[\hat{\mathbf{V}}_{bc}(t'), \hat{\boldsymbol{\rho}}_{bc}(0) \right]^{T_b} \\
&= \left\langle \hat{\mathbf{V}}_a \right\rangle \left(\left[\hat{\mathbf{V}}_b(t'), \hat{\boldsymbol{\rho}}_b \right]^T \otimes \hat{\boldsymbol{\rho}}_c + \hat{\boldsymbol{\rho}}_b^T \otimes \left[\hat{\mathbf{V}}_c(t'), \hat{\boldsymbol{\rho}}_c \right] \right) \\
&= - \left\langle \hat{\mathbf{V}}_a \right\rangle \left(\left[\hat{\mathbf{V}}_b(t')^T, \hat{\boldsymbol{\rho}}_b \right] \otimes \hat{\boldsymbol{\rho}}_c - \hat{\boldsymbol{\rho}}_b \otimes \left[\hat{\mathbf{V}}_c(t'), \hat{\boldsymbol{\rho}}_c \right] \right),
\end{aligned}
\tag{39}
$$

where $\left\langle \hat{\mathbf{V}}_a \right\rangle$ means the thermal average of the operator $\hat{\mathbf{V}}_a$ and the notation $\hat{\mathbf{V}}_{bc} \equiv \hat{\mathbf{V}}_b \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \hat{\mathbf{V}}_c$ is introduced. The initial condition $\hat{\boldsymbol{\rho}}_{bc}(0) = \hat{\boldsymbol{\rho}}_b \otimes \hat{\boldsymbol{\rho}}_c$ was used. Since $\hat{\boldsymbol{\rho}}_{b,c} |i\rangle = \delta_{i,1} |i\rangle$ the term Eq. (39) does not contribute to the eigenvalues of the matrix (37) in the first order.

To simplify the calculation of the second order corrections it is assumed that the thermal average of the system A coupling operator $\left\langle \hat{\mathbf{V}}_a \right\rangle$ vanishes. This assumption is not crucial for the qualitative picture of temperature dependence of the entanglement. Moreover, it is in line with common models of coupling, for example, the Caldeira-Leggett model [31], dipole interaction with the electromagnetic field [32], etc. The integrand in the second order term in Eq. (33) is:

$$
\begin{aligned}
Tr_a \left(\left[\hat{\mathbf{V}}(t'), \left[\hat{\mathbf{V}}(t''), \hat{\boldsymbol{\rho}}(0) \right] \right] \right)^{T_b} &= \left\langle \hat{\mathbf{V}}_a(t') \hat{\mathbf{V}}_a(t'') \right\rangle \left[\hat{\mathbf{V}}_{bc}(t'), \hat{\mathbf{V}}_{bc}(t'') \hat{\boldsymbol{\rho}}_{bc} \right]^{T_b} \\
&- \left\langle \hat{\mathbf{V}}_a(t'') \hat{\mathbf{V}}_a(t') \right\rangle \left[\hat{\mathbf{V}}_{bc}(t'), \hat{\boldsymbol{\rho}}_{bc} \hat{\mathbf{V}}_{bc}(t'') \right]^{T_b}.
\end{aligned}
\tag{40}
$$

Expanding the thermal averages in the orthonormal basis $|n\rangle$ of the Hamiltonian H_a leads to:

$$
\begin{aligned}
Tr_a \left(\left[\hat{\mathbf{V}}(t'), \left[\hat{\mathbf{V}}(t''), \hat{\boldsymbol{\rho}}(0) \right] \right] \right)^{T_b} &= \sum_{m,n} p_{a,n} \left| \left\langle m \left| \hat{\mathbf{V}}_a \right| n \right\rangle \right|^2 (\cos(\omega_{mn}^a (t' - t'')) \left[\hat{\mathbf{V}}_{bc}(t'), \left[\hat{\mathbf{V}}_{bc}(t''), \hat{\boldsymbol{\rho}}_{bc} \right] \right] \\
&+ i \sin(\omega_{mn}^a (t' - t'')) \left[\hat{\mathbf{V}}_{bc}(t'), \left\{ \hat{\mathbf{V}}_{bc}(t''), \hat{\boldsymbol{\rho}}_{bc} \right\} \right])^{T_b},
\end{aligned}
\tag{41}
$$

where ω_{mn}^a is the energy difference between the states $|n\rangle$ and $|m\rangle$ of the Hamiltonian H_a, the $\left\{ \hat{\mathbf{X}}, \hat{\mathbf{Y}} \right\}$ designates anticommutator of operators $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ and $p_{a,n} \equiv (\hat{\boldsymbol{\rho}}_a)_{nn}$.

For simplicity the notation $\hat{\mathbf{C}}(t', t'')$ is used for the operator (41). Expressing the operator $\hat{\mathbf{V}}_{bc}$ in terms of $\hat{\mathbf{V}}_b$ and $\hat{\mathbf{V}}_c$ we put the matrix elements of $\hat{\mathbf{C}}(t', t'')$ into the following form:

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$$\langle 12| \hat{\mathbf{C}}(t',t'') |12\rangle = -2P_{11} \sum_{m,n} p_{a,n} \left| \left\langle m \left| \hat{\mathbf{V}}_a \right| n \right\rangle \right|^2 \left| \left\langle 1 \left| \hat{\mathbf{V}}_c \right| 2 \right\rangle \right|^2 \cos((\omega_{mn}^a + \omega_c)(t' - t''))$$

$$\langle 21| \hat{\mathbf{C}}(t',t'') |21\rangle = -2P_{11} \sum_{m,n} p_{a,n} \left| \left\langle m \left| \hat{\mathbf{V}}_a \right| n \right\rangle \right|^2 \left| \left\langle 1 \left| \hat{\mathbf{V}}_b \right| 2 \right\rangle \right|^2 \cos((\omega_{mn}^a + \omega_b)(t' - t'')) \quad (42)$$

$$\langle 12| \hat{\mathbf{C}}(t',t'') |21\rangle = P_{11} \sum_{m,n} p_{a,n} \left| \left\langle m \left| \hat{\mathbf{V}}_a \right| n \right\rangle \right|^2 \left\langle 2 \left| \hat{\mathbf{V}}_b \right| 1 \right\rangle \left\langle 2 \left| \hat{\mathbf{V}}_c \right| 1 \right\rangle e^{i\omega_{mn}^a(t'-t'')}(e^{-i(\omega_b t' + \omega_c t'')}$$

$$+ e^{-i(\omega_b t'' + \omega_c t')}),$$

where $\omega_{b,c}$ stands for the energy difference between the first excited and the ground states of the unperturbed subsystem B (C). The matrix elements $M_{12,12}$, $M_{21,21}$ and $M_{12,21}$ are given by:

$$M_{12,12} = -\gamma^2 \int_0^t \int_0^{t'} \langle 12| C(t',t'') |12\rangle \, dt' dt''.$$

$$M_{21,21} = -\gamma^2 \int_0^t \int_0^{t'} \langle 21| C(t',t'') |21\rangle \, dt' dt'' \qquad (43)$$

$$M_{12,21} = -\gamma^2 \int_0^t \int_0^{t'} \langle 12| C(t',t'') |21\rangle \, dt' dt''.$$

The integration is straightforward but the final expressions are cumbersome. Two cases are considered explicitly: (a) $\omega_{mn}^a \gg \omega_{b,c}$ and (b) $\omega_{b,c} \gg \omega_{mn}^a$. In both cases it is shown that at sufficiently low initial temperature of the system $B \otimes C$ one of the eigenvalues of the matrix (37) is negative and the lower bound temperature T_{lb} is calculated.

A. Two "slow" systems interacting with a "fast" common third party

Performing the integrations in Eq. (43) and taking the leading terms in $\omega_{b,c}/\omega_{mn}^a$ brings to:

$$M_{12,12} = 4\gamma^2 \sum_{m,n} p_{a,n} \left| \left\langle m \left| \hat{\mathbf{V}}_a \right| n \right\rangle \right|^2 \left| \left\langle 1 \left| \hat{\mathbf{V}}_c \right| 2 \right\rangle \right|^2 \frac{\sin((\omega_{mn}^a + \omega_c)t/2)^2}{(\omega_{mn}^a)^2}$$

$$M_{21,21} = 4\gamma^2 \sum_{m,n} p_{a,n} \left| \left\langle m \left| \hat{\mathbf{V}}_a \right| n \right\rangle \right|^2 \left| \left\langle 1 \left| \hat{\mathbf{V}}_b \right| 2 \right\rangle \right|^2 \frac{\sin((\omega_{mn}^a + \omega_b)t/2)^2}{(\omega_{mn}^a)^2} \qquad (44)$$

$$M_{12,21} = 2\gamma^2 \sum_{m,n} p_{a,n} \left| \left\langle m \left| \hat{\mathbf{V}}_a \right| n \right\rangle \right|^2 \left\langle 2 \left| \hat{\mathbf{V}}_b \right| 1 \right\rangle \left\langle 2 \left| \hat{\mathbf{V}}_c \right| 1 \right\rangle \frac{(1 - e^{-i(\omega_b + \omega_c)t})}{\omega_{mn}^a(\omega_b + \omega_c)}.$$

At $T = 0$ the minimal eigenvalue of Eq. (38) is given by $\lambda_- = -\sqrt{M_{12,12}M_{21,21} - |M_{12,21}|^2}$, which to the leading order in $\omega_{b,c}/\omega_{mn}^a$ gives $\lambda_- = -|M_{12,21}|^2$. This proves that the system $B \otimes C$ becomes entangled at sufficiently low temperature. We note that this result holds at any finite temperature of the system A. At infinite temperature of the system A $M_{12,21} \equiv 0$ and no free entanglement is generated in the system $B \otimes C$.

At finite initial temperature of $B \otimes C$ the condition $\lambda_- < 0$ translates to $P_{12}P_{21} < \gamma^4 |M_{12,21}|^2 P_{11}^2$ to the leading order in $\omega_{1,2}/\omega_{mn}^a$. The lower bound temperature T_{lb} is found from the condition $P_{12}P_{21} = \gamma^4 |M_{12,21}|^2 P_{11}^2$. Since $|M_{12,21}|$ is an oscillating function of time the amplitude of $|M_{12,21}|$ must be substituted for $|M_{12,21}|$ in this equality, which leads to the following equation defining the lower bound temperature:

$$4\gamma^2 \frac{\left|\left\langle 2 \left| \hat{\mathbf{V}}_b \right| 1 \right\rangle \left\langle 2 \left| \hat{\mathbf{V}}_c \right| 1 \right\rangle\right|}{\omega_b + \omega_c} \sum_{m,n} \frac{p_{a,n} \left|\left\langle m \left| \hat{\mathbf{V}}_a \right| n \right\rangle\right|^2}{\omega_{mn}^a} = \sqrt{\frac{P_2 P_3}{P_1^2}}$$

$$= \exp\left(-\frac{\omega_b + \omega_c}{2T_{lb}}\right), \tag{45}$$

finally leading to:

$$T_{lb} = \frac{-(\omega_b + \omega_c)}{2\ln\left(4\gamma^2 \frac{|\langle 2|\hat{\mathbf{V}}_b|1\rangle\langle 2|\hat{\mathbf{V}}_c|1\rangle|}{\omega_b + \omega_c} \sum_{m,n} \frac{p_{a,n}|\langle m|\hat{\mathbf{V}}_a|n\rangle|^2}{\omega_{mn}^a}\right)}. \tag{46}$$

A generalization of the formula to the case of interaction of the form $\sum \gamma_i \hat{\mathbf{V}}_a^i \otimes (\hat{\mathbf{V}}_b^i \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \hat{\mathbf{V}}_c^i)$ can be carried out along the same lines.

The entanglement in the reduced system of two noninteracting "slow" spins interacting with the "fast" four level "bath" was explored numerically and the results are plotted on Fig. 6. The shaded area in the parametric space of the logarithm of inverse coupling strength and the inverse initial temperature of the spins represents parametric values for which no entanglement develops in the course of the evolution. The border of the shaded area corresponds to the critical temperature for various coupling magnitudes. The Hamiltonian of the composite system is:

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_a \otimes \hat{\mathbf{1}}_b \otimes \hat{\mathbf{1}}_c + \frac{1}{2}\omega(\hat{\mathbf{1}}_a \otimes (\hat{\boldsymbol{\sigma}}_z^b \otimes \hat{\mathbf{1}}_c + \sqrt{2}\hat{\mathbf{1}}_b \otimes \hat{\boldsymbol{\sigma}}_z^c)) \tag{47}$$
$$+ \gamma\hat{\mathbf{V}}_a \otimes (\hat{\boldsymbol{\sigma}}_x^b \otimes \hat{\mathbf{1}}_c + \hat{\mathbf{1}}_b \otimes \hat{\boldsymbol{\sigma}}_x^c),$$

where $(\hat{\mathbf{H}}_a)_{ij} = \delta_{ij}E_a^i$, $E_a^{\{1,2,3,4\}} = \{0, 10\omega, 20\omega, 30\omega\}$ and $(\hat{\mathbf{V}}_a)_{ij} = \delta_{ij}$. The temperature of the thermal initial state of the "bath" is $T = 5\omega$. The value of ω chosen for the numerical
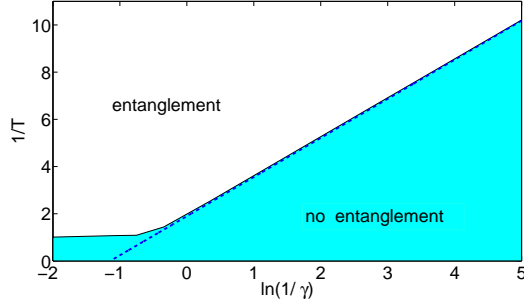
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FIG. 6: The shaded area in the parameter space of the inverse initial temperature T of the "slow" spins and the logarithm of the inverse coupling strength γ, represents values of T and γ where entanglement does not develop in the course of the evolution. The composite system of two "slow" spins interacting with a "fast" four level system evolves from the initial product of thermal states under the Hamiltonian (47). The dashed line is the plot of T_{lb}, Eq. (46). Up to the coupling $\gamma = 1$ its correspondence to the border of the shaded area is very good.

calculation is unity. The correspondence of Eq. (46) (the dashed line) to the numerical values is very good up to a coupling strength of the order of unity. We note that for large values of the coupling strength γ the critical temperature asymptotically tends to a finite constant value.

B. Two "fast" systems interacting with a "slow" common third party

The case $\omega_{b,c} >> \omega_{mn}^a$ is more complex. To demonstrate entanglement at zero temperature of the system $B \otimes C$ two simplifying assumptions were added. The first is that the temperature of the system A is also zero. The second is that the matrix elements of V_a couple only the neighboring states: $\langle n | \hat{\mathbf{V}}_a | m \rangle \propto \delta_{n,m\pm1}$. Under these two assumptions the

expressions for $M_{12,12}$, $M_{21,21}$ and $M_{12,12}M_{M_{21,21}} - |M_{12,21}|^2$ become:

$$M_{12,12} = P_{11} \left(\frac{2\gamma \left|\left\langle 2 \left| \hat{\mathbf{V}}_a \right| 1 \right\rangle\right| \left|\left\langle 1 \left| \hat{\mathbf{V}}_c \right| 2 \right\rangle\right| \sin\left(\frac{(\omega_a+\omega_c)t}{2}\right)}{\omega_c} \right)^2$$

$$M_{21,21} = P_{11} \left(\frac{2\gamma \left|\left\langle 2 \left| \hat{\mathbf{V}}_a \right| 1 \right\rangle\right| \left|\left\langle 1 \left| \hat{\mathbf{V}}_b \right| 2 \right\rangle\right| \sin\left(\frac{(\omega_a+\omega_b)t}{2}\right)}{\omega_b} \right)^2$$

$$M_{12,12}M_{M_{21,21}} - |M_{12,21}|^2 \tag{48}$$

$$= P_{11}^2 \left(\frac{2\gamma^2 \left|\left\langle 2 \left| \hat{\mathbf{V}}_a \right| 1 \right\rangle\right|^2 \left|\left\langle 1 \left| \hat{\mathbf{V}}_c \right| 2 \right\rangle\right| \left|\left\langle 1 \left| \hat{\mathbf{V}}_b \right| 2 \right\rangle\right|}{\omega_b \omega_c} \right)^2 S(t),$$

where

$$S(t) = \sin(\omega_a t)[\sin(\omega_b t) + \sin(\omega_c t)$$

$$- \sin((\omega_a + \omega_b + \omega_c)t)]. \tag{49}$$

To estimate $S(t)$ new variables $x = \sin(\omega_a t)$, $y = \sin(\omega_b t)$ and $z = \sin(\omega_c t)$ are introduced. Ignoring the zero measure set of commensurable frequencies we can treat the function $S(t)$ as function of three independent variables x, y and z. The range of $S(t)$ in the cube, defined by $-1 \le x, y, z \le 1$, can be explored numerically and is found to be: $s \le S(t) \le 3$, where $s \approx -1.6834$. Therefore, from Eq.(48) $M_{12,12}M_{M_{21,21}} - |M_{12,21}|^2 < 0$, which proves that at zero temperature $\lambda_- < 0$ (Cf. Eq.(38)) and the systems B and C are entangled by the interaction with the system A.

The lower bound temperature is determined by the condition $\lambda_- = 0$, which translates to $(P_{12} + M_{12,12})(P_{21} + M_{21,21}) = |M_{12,21}|^2$ (Cf. Eq.(38)). The latter condition can be put in the form $(M_{12,12}M_{21,21} - |M_{12,21}|^2) + P_{12}P_{21} + P_{12}M_{21,21} + P_{21}M_{12,12} = 0$. Since $M_{12,21}$ and $M_{21,21}$ are nonnegative independent functions of time the minimum value of $(M_{12,12}M_{21,21} - |M_{12,21}|^2) + P_{12}P_{21} + P_{12}M_{21,21} + P_{21}M_{12,12}$ is obtained at $M_{21,21} = M_{12,12} = 0$. Then the lower bound temperature can be calculated from the condition that the amplitude of $M_{12,12}M_{21,21} - |M_{12,21}|^2$ equals $-P_{12}P_{21}$:

$$\frac{2\gamma^2 \sqrt{|s|} \left|\left\langle 2 \left| \hat{\mathbf{V}}_a \right| 1 \right\rangle\right|^2 \left|\left\langle 1 \left| \hat{\mathbf{V}}_c \right| 2 \right\rangle\right| \left|\left\langle 1 \left| \hat{\mathbf{V}}_b \right| 2 \right\rangle\right|}{\omega_b \omega_c} = \sqrt{\frac{P_{12}P_{21}}{P_{11}^2}}, \tag{50}$$

finally leading to:

$$T_{lb} = \frac{-(\omega_b + \omega_c)}{2\ln\left(2\gamma^2\sqrt{|s|}\frac{|\langle 2|\hat{\mathbf{V}}_a|1\rangle|^2|\langle 1|\hat{\mathbf{V}}_c|2\rangle||\langle 1|\hat{\mathbf{V}}_b|2\rangle|}{\omega_b\omega_c}\right)}. \tag{51}$$

It is interesting to note that T_{lb} in this case does not depend on the time scales of the "slow" system.

The entanglement in the reduced system of two noninteracting "fast" spins interacting with the "slow" four level "bath" was explored numerically and the results are plotted on Fig. 7. The shaded area in the parametric space of the logarithm of inverse coupling strength and the inverse initial temperature of the spins represents parametric values for which no entanglement develops in the course of the evolution. The border of the shaded area corresponds to the critical temperature for various coupling magnitudes. The Hamiltonian is chosen to be similar to the previous example, Cf. Eq.(47), but time scales of the subsystems are reversed:

$$\begin{aligned}\hat{\mathbf{H}} = {} & \hat{\mathbf{H}}_a \otimes \hat{\mathbf{1}}_b \otimes \hat{\mathbf{1}}_c + 5\omega(\hat{\mathbf{1}}_a \otimes (\hat{\boldsymbol{\sigma}}_z^b \otimes \hat{\mathbf{1}}_c + \sqrt{2}\hat{\mathbf{1}}_b \otimes \hat{\boldsymbol{\sigma}}_z^c)) \\ & + \gamma\hat{\mathbf{V}}_a \otimes (\hat{\boldsymbol{\sigma}}_x^b \otimes \hat{\mathbf{1}}_c + \hat{\mathbf{1}}_b \otimes \hat{\boldsymbol{\sigma}}_x^c),\end{aligned} \tag{52}$$

where $(\hat{\mathbf{H}}_a)_{ij} = \delta_{ij}E_a^i$, $E_a^{\{1,2,3,4\}} = \{0, \omega, 2\omega, 3\omega\}$ and $(\hat{\mathbf{V}}_a)_{ij} = \delta_{ij}$. The temperature of the thermal initial state of the "bath" was chosen as $T = 0.01\omega$, which is small compared to the energy scale of the "bath" chosen for the numerical calculation: $\omega = 1$. The dashed line on the Fig. 7 is a plot of Eq. (51) and the correspondence to the border of the shaded area at coupling strength up to the order of unity is good.

IV. SUMMARY AN CONCLUSIONS

Entanglement is created by both direct and indirect weak interaction between two initially disentangled systems prepared in thermal states at sufficiently low temperatures. The study is restricted to the conditions where the ground states of both systems are not invariant under the interaction and the interaction is nonresonant. As a consequence, the present analysis left out some interesting models such as the Jaynes-Cummings model [33]. The Jaynes-Cummings model of interacting two level system and a quantized field mode was investigated in Ref.[16]. It was found that no free entanglement is generated in the course
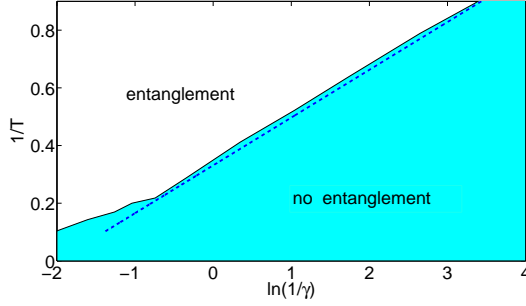
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FIG. 7: The shaded area in the parameter space of the inverse initial temperature T of the "fast" spins and the logarithm of the inverse coupling strength γ, represents values of T and γ where entanglement does not develop in the course of the evolution. The composite system of two "fast" spins interacting with the "slow" four level system evolves from the initial product of thermal states under the Hamiltonian (52). The dashed line is the plot of T_{lb}, Eq. (51). Up to the coupling $\gamma = 1$ its correspondence to the border of the shaded area is good.

of the evolution of the composite system if the initial temperature of both the subsystems is sufficiently high.

The generation of entanglement in cases of the weak *resonant* direct and undirect interactions will be treated separately [34].

In the case of indirect interaction to show entanglement at $T = 0$ we have assumed that the thermal average of the third party coupling term in the initial state vanishes. The reason for the assumption was technical. It should be noted that many system-bath models of linear coupling satisfy this assumption [31]. The additional technical assumption was that the coupling terms of the noninteracting parties possess matrix elements only between the adjacent energy states. Here, too, the assumption is general for weak coupling models. Two cases of time scale separation were considered explicitly. The first is the case of two "slow" systems interacting via the "fast" third common party. The second is the case of two "fast" systems interacting via the "slow" third common party. In the first case the entanglement was shown to appear at sufficiently low initial temperature of the "slow" systems for any finite temperature of the third party. In the second case the entanglement develops at sufficiently low initial temperature of the "fast" systems. In this case we assumed that the third party was prepared at zero temperature and that the third party coupling agent has nonvanishing matrix elements only between the adjacent energy states. This assumption is

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stronger than just assuming that its thermal average vanishes.

In these cases of indirect interaction and in the case of the direct interaction between the parts we have shown that if the initial temperature of the bipartite state is zero entanglement is generated by the interaction. At sufficiently high temperature the composite state remains PPT in the course of evolution. From these results it follows that a lower critical temperature T_{lc} exists: if the initial temperature of both thermal states is below T_{lc} the interaction generates entanglement in the course of the evolution, and if the initial temperature is sufficiently close to T_{lc} from above the the composite state remains PPT forever. When the composite system is finite dimensional there exists an upper critical temperature T_{uc}: if the initial temperature of both thermal states is higher than T_{uc} the composite state remains PPT in the course of evolution and if the initial temperature is sufficiently close to T_{uc} from below entanglement is generated. We conjecture on the basis of numerical experiments that $T_{lc} = T_{uc}$ in general. In both cases of a direct and an indirect interaction between the initially disentangled systems, prepared in thermal states, we calculated the lower bound T_{lb} for the lower critical temperature T_{lc}. When the initial temperature of both thermal states is below T_{lb} the interaction generates entanglement in the course of the evolution. For temperatures above the lower bound T_{lb} the negativity of the partially transposed composite state is zero in the leading order in the coupling strength and therefore negligible in the weak coupling limit. It follows, that T_{lb} may be considered as the *physical* critical temperature for the negativity of the composite state.

Separable states can be considered as classical states, because they lack quantum correlations. One may hope that, as a consequence, the dynamics of separable states can be efficiently simulated on classical computers. Whether this is possible is an open question in quantum information science. If a moderate scaling procedure exists for the simulation of the dynamics of a separable bipartite state, then it seems that such a procedure exists also if the evolving state remains PPT for all times. Ref.[26] has proved that a density operator $\hat{\rho}$ supported on a $M \times N$ dimensional Hilbert space ($M \leq N$) with positive partial transpose and a rank smaller than or equal to N is separable. It follows that a PPT state of dimension N is always separable when embedded in the larger Hilbert space of dimension N^2 or higher. The dynamics of the low dimensional PPT state will be physically equivalent to the dynamics of the high dimensional separable state which can (hopefully) be simulated efficiently on the classical computer.

The present analysis shows that above a critical temperature T_{lb} the PPT character of a composite state is preserved along the evolution. The challenge is to construct an effective simulation for the dynamics of a composite quantum systems at finite temperature employing classically based computers.

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